Quantitative estimates for the effect of disorder on low-dimensional lattice systems

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Lattice systems with compact state space

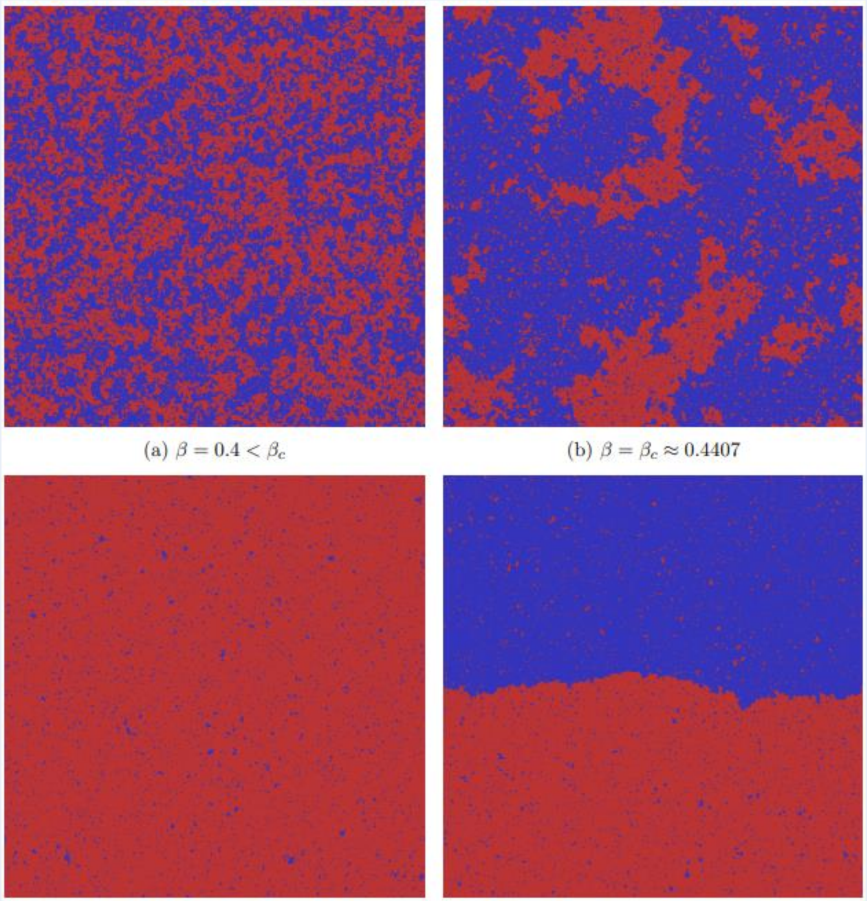
- We discuss statistical physics systems on $\mathbb{Z}^d$, aiming to develop a quantitative understanding of the effect of adding disorder to them.
- We start with the case of a compact state space.
- Setup: (1) Compact metric space S equipped with a Borel measure κ.
  (2) Translation-invariant finite range and finite energy Hamiltonian H.
- As usual, for a finite domain $\Lambda \subset \mathbb{Z}^d$, at temperature T and with boundary conditions $\tau: \mathbb{Z}^d \to S$, configurations $\sigma: \mathbb{Z}^d \to S$ coinciding with τ outside Λ are sampled from the probability measure with density

$$\frac{1}{Z_{T,\Lambda,\tau}} \exp\left(-\frac{1}{T} H_\Lambda(\sigma)\right)$$

  with respect to the measure $\prod_v d\kappa(\sigma_v)$, where $Z_{T,\Lambda,\tau}$ is the partition function and H_Λ contains the terms in the Hamiltonian depending on the spins in Λ.
  Periodic boundary conditions and the zero-temperature limit are also allowed.
- Examples: Ising model: $S = \{-1,1\}$, $\kappa = \text{counting}$, $H(\sigma) = -\sum_{u\sim v} \sigma_u \sigma_v$
- Potts model: $S = \{1,2,\dots,q\}$, $\kappa = \text{counting}$, $H(\sigma) = -\sum_{u\sim v} 1_{\sigma_u=\sigma_v}$
- Spin O(n) model with $n \geq 2$: $S = \mathbb{S}^{n-1}$, $\kappa = \text{uniform}$, $H(\sigma) = \sum_{u\sim v} |\sigma_u - \sigma_v|^2$

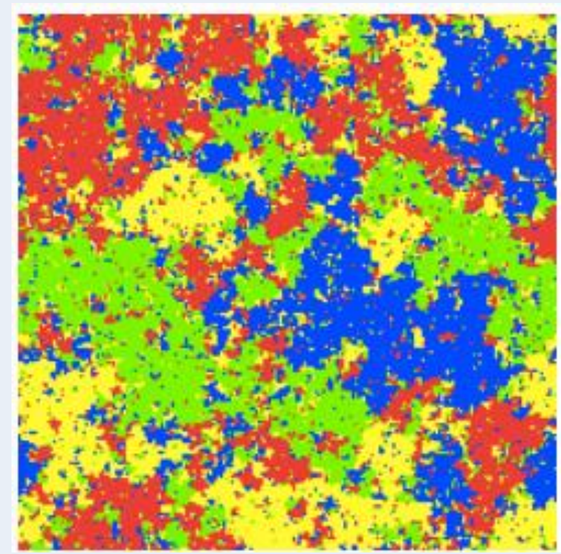
Phase transitions in pure systems I

- Ising model: $S = \{-1, 1\}$, $\kappa =$ counting, $H(\sigma) = -\sum_{u \sim v} \sigma_u \sigma_v$.
- The Ising model undergoes a phase transition in dimensions $d \geq 2$ as the temperature is lowered, from a disordered to an ordered state.
- Similar behavior for the q-state Potts model ($S = \{1, 2, \ldots, q\}$, $H(\sigma) = -\sum_{u \sim v} 1_{\sigma_u = \sigma_v}$).



(a) $\beta = 0.4 < \beta_c$ (b) $\beta = \beta_c \approx 0.4407$ (c) $\beta = 0.5 > \beta_c$ (d) $\beta = 0.5$ with Dobrushin boundary conditions

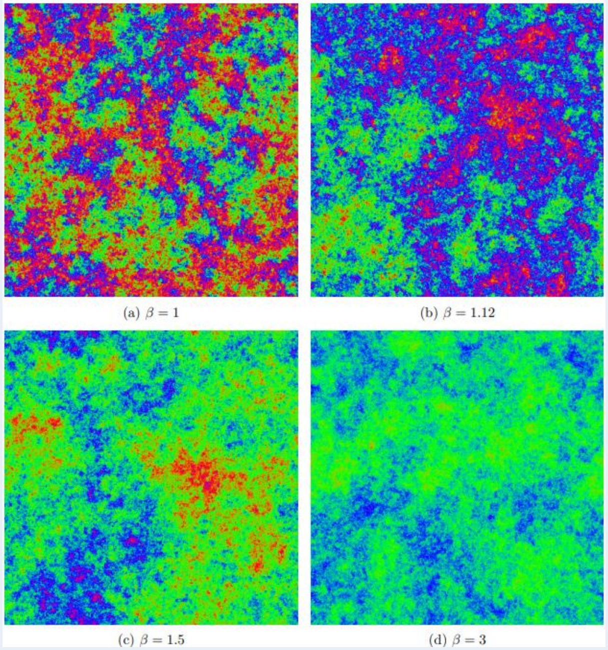
Simulation from Spinka–Peled 2019



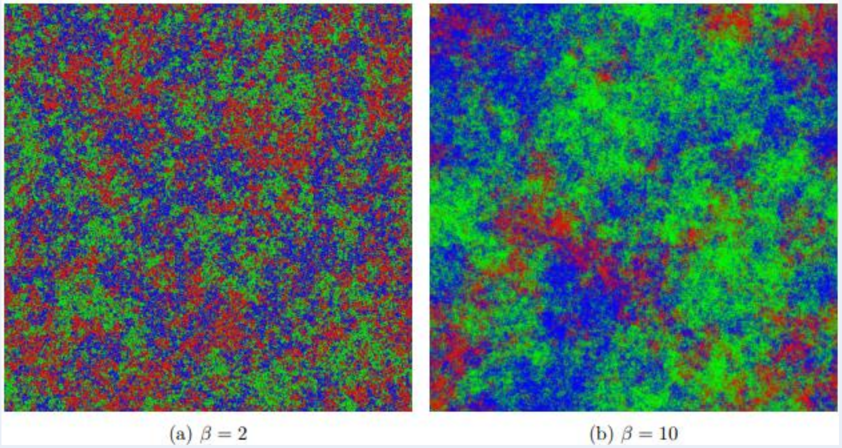
d=2 Potts model with q=4 at criticality.
Simulation by Beffara.

Phase transitions in pure systems II

- Spin O(n) model with $n \geq 2$: $S = \mathbb{S}^{n-1}, \kappa =$ uniform, $H(\sigma) = \sum_{u\sim v} |\sigma_u - \sigma_v|^2$
- Mermin–Wagner theorem: The spin O(n) model does not exhibit an ordered phase in two dimensions (even at low temperature).
- Fröhlich–Simon–Spencer theorem: A low-temperature ordered phase exists in dimensions $d \geq 3$.
- In two dimensions:
  n=2 (XY model): Berezinskii–Kosterlitz–Thouless transition (proof by Fröhlich–Spencer) from exponential to power-law decay of correlations as temperature is lowered.
  n=3 (Heisenberg model): Polyakov conjecture – exponential decay at all temperatures.



XY model simulation from Spinka–Peled 2019



Heisenberg model simulation from Spinka–Peled 2019

Disordered lattice systems

- Noised observables: Let $f: S^{\mathbb{Z}^d} \to \mathbb{R}^m$, for some $m \geq 1$, be a bounded measurable function depending on the spins in a finite neighborhood of the origin.
  Disorder: Let $(\eta_v)_{v\in\mathbb{Z}^d}$ be independent standard m-dimensional Gaussian vectors.
  Disordered Hamiltonian: $H^\eta(\sigma) = H(\sigma) - \lambda \sum_v \eta_v \cdot f\big(\mathcal{T}_v(\sigma)\big)$
  where $\mathcal{T}_v(\sigma)$ is the configuration σ translated by v.
- Examples: Random-field Ising model: $m = 1$ and $f(\sigma) = \sigma_0$. Thus

$$H^\eta(\sigma) = -\sum_{u\sim v} \sigma_u\sigma_v - \lambda \sum_v \eta_v\sigma_v$$

- Edwards-Anderson spin glasses: $S = \{-1,1\}, \mu = \text{counting}, f(\sigma) = \left(\sigma_{e_j}\sigma_{\mathbf{0}}\right)_{j=1}^{d}$.

$$H^\eta(\sigma) = -\lambda \sum_{u\sim v} \eta_{u,v}\sigma_u\sigma_v$$

- Random-field q-state Potts model: $\mathrm{m} = \mathrm{q}$ and $f(\sigma) = \left(1_{\sigma_{\mathbf{0}}=1}, \ldots, 1_{\sigma_{\mathbf{0}}=q}\right)$. Thus

$$H^\eta(\sigma) = -\sum_{u\sim v} 1_{\sigma_u=\sigma_v} - \lambda \sum_v \sum_{k=1}^{q} \eta_{v,k} 1_{\sigma_v=k}$$

- Random-field spin $O(n)$ model, $n \geq 2$: $m = n$ and $f(\sigma) = \sigma_0$ (with $\mathbb{S}^{\mathrm{n}-1} \subset \mathbb{R}^n$),

$$H^\eta(\sigma) = \sum_{u\sim v} |\sigma_u - \sigma_v|^2 - \lambda \sum_v \eta_v \cdot \sigma_v$$

Imry-Ma phenomenon

- Imry-Ma (1975) considered the effects of disorder for the random-field Ising and spin $O(n)$ models, and predicted that in low dimensions, an arbitrarily small disorder strength λ causes the models to lose their ordered phase, as follows:
  The random-field Ising model is disordered at all temperatures for $d \leq 2$.
  The random-field spin $O(n)$ model is disordered at all temperatures for $d \leq 4$.
- Aizenman-Wehr (1989) proved the predictions as part of a general statement.
- Notation: Write $\Lambda_L := \{-L, \dots, L\}^d$. For each disorder η, write $\langle\cdot\rangle_\mu$ for the thermal expectation according to a Gibbs measure μ of the η-disordered system.
  Write $\mathbb{P}$ and $\mathbb{E}$ for the probability and expectation operator over η.
- Theorem (Aizenman-Wehr, special case): For a disordered lattice system with compact state space (as discussed above) in dimensions $d = 1, 2$, at temperature $0 \leq T < \infty$ and disorder strength $\lambda > 0$, the limit
  $$\lim_{L\to\infty} \frac{1}{L^d} \sum_{v \in \Lambda_L^d} \langle f\big(\mathcal{T}_v(\sigma)\big)\rangle_\mu$$
  exists and has the same value for all Gibbs measures μ and almost all η.
  The same holds in dimensions $1 \leq d \leq 4$ for the spin $O(n)$ models with $n \geq 2$.
- Our goal: Develop a quantitative understanding of this phenomenon.

Random-field Ising model

- Random-field Ising model Hamiltonian: $H^\eta(\sigma) = -\sum_{u\sim v} \sigma_u\sigma_v - \lambda \sum_v \eta_v\sigma_v$
- The disordered model still satisfies the usual monotonicity (FKG) properties.
  In particular, the model has maximal and minimal Gibbs measures $\mu^{\eta,+}$ and $\mu^{\eta,-}$, arising in the thermodynamic limit from constant boundary conditions. The Aizenman-Wehr theorem implies that $\mu^{\eta,+} = \mu^{\eta,-}$ in two dimensions η-almost surely, so that the model has a unique Gibbs measure.
- A natural quantitative parameter is $m_L := \mathbb{E}(\langle\sigma_0\rangle^+_{\Lambda_L})$ where $\langle\cdot\rangle^+_{\Lambda_L}$ denotes the thermal expectation in $\{-L,\dots,L\}^2$ with $+1$ boundary conditions.
- A bound of the form $m_L \le \exp(-c(\lambda,T)L)$ is relatively simple for large disorder strength λ or high temperatures T, so interested in small λ and low temperature.
- Results: In dimension $d = 2$: $m_L \le \frac{C(\lambda)}{\sqrt{\log\log L}}$ (Chatterjee 2017), $m_L \le \frac{C(\lambda)}{L^{c(\lambda)}}$ (Aizenman-P. 2018) and finally

$$m_L \le C(\lambda)\exp\left(-\frac{L}{\ell(\lambda)}\right)$$

  proved at zero temperature by Ding-Xia 2019 and then at positive temperature by Ding-Xia 2019 and Aizenman-Harel-P. 2019.

- Still open to determine correlation length $\ell(\lambda)$ for small λ. Proof implies $\ell(\lambda) \le e^{e^{1/\lambda^2}}$ (Bar-Nir 2022). Ding-Wirth (2020): Correlation length $= e^{\Theta(\lambda^{-\frac{4}{3}})}$ in another sense.

Random-field Ising and Potts models

- Dimension $d \geq 3$, weak disorder (small λ): Imbrie 1985 (zero temperature) and Bricmont-Kupiainen 1988 (all temperatures) established long-range order in the random-field Ising model. A shorter argument was given recently by Ding-Zhuang (2021), also extending the result to the random-field Potts model.
- Ding-Liu-Xia (2022), making use of Ding-Song-Sun (2023), extend the long-range order result to all temperatures lower than the critical temperature of non-disordered Ising model. Ding-Huang-Xia (2023) investigate the critical scaling for the disorder at the critical temperature of the non-disordered Ising model.
- Rigas (2022) extended part of the correlation length result of Ding-Wirth to the random-field Potts model.

Quantitative results

- The other models discussed (Potts, spin-glasses, spin $O(n)$) do not share the monotonicity properties of the random-field Ising model and the proof techniques break down for them. Indeed, even the choice of which quantity to bound is non-obvious since it is unclear which boundary conditions τ maximize or minimize the average $\langle f(\mathcal{T}_v(\sigma))\rangle^{\tau}_{\Lambda_L^2}$ and, indeed, it may be that these boundary conditions depend on the disorder η and on L and v. We obtain the following results.

- Theorem (Dario-Harel-P 2020+): For each two-dimensional disordered lattice system of the type described above, at temperature $0 \le T < \infty$ and disorder strength $\lambda > 0$, there exists $C > 0$ so that for all $L \ge 2$,

$$\mathbb{E}\left(\sup_{\tau_1,\tau_2:\mathbb{Z}^2\to S}\left\|\frac{1}{L^2}\sum_{v\in\Lambda_L^2}\langle f(\mathcal{T}_v(\sigma))\rangle^{\tau_1}_{\Lambda_L^2} - \langle f(\mathcal{T}_v(\sigma))\rangle^{\tau_2}_{\Lambda_L^2}\right\|\right) \le \frac{C}{(\log\log L)^{\frac{1}{4}}}$$

For the d-dimensional random-field spin $O(n)$ model with $n \ge 2$, at temperature $0 \le T < \infty$ and disorder strength $\lambda > 0$, there exists $C > 0$ so that for all $L \ge 2$,

$$\mathbb{E}\left(\sup_{\tau:\mathbb{Z}^d\to S}\left\|\frac{1}{L^d}\sum_{v\in\Lambda_L^d}\langle \sigma_v\rangle^{\tau}_{\Lambda_L^d}\right\|\right) \le C\begin{cases} L^{-\frac{1}{3}} & d=2 \\ L^{-\frac{1}{5}} & d=3 \\ (\log\log L)^{-\frac{1}{2}} & d=4 \end{cases}$$

Uniqueness problem

- Conjecture: For a disordered lattice system with compact state space (as discussed above) in dimension $d = 2$, at temperature $0 \leq T < \infty$ and disorder strength $\lambda > 0$, it holds that η-almost surely, for all vertices $v \in \mathbb{Z}^2$, the value of
$$\langle f\big(\mathcal{T}_v(\sigma)\big)\rangle_\mu$$
is the same for all Gibbs measures μ of the η-disordered system.

- The conjecture is equivalent to the following finite-volume statement:
$$\lim_{L\to\infty} \sup_{\tau_1,\tau_2:\mathbb{Z}^2\to S} \left\| \langle f(\sigma)\rangle_{\Lambda_L^2}^{\tau_1} - \langle f(\sigma)\rangle_{\Lambda_L^2}^{\tau_2} \right\| = 0, \qquad \eta\text{−almost surely}$$

- The value of $\mathcal{T}_v(\sigma)$ itself need not be unique in general systems.
For instance, a global sign flip applied to σ in a spin glass system (with Hamiltonian $H^\eta(\sigma) = -\lambda \sum_{u\sim v} \eta_{u,v}\sigma_u\sigma_v$) takes one Gibbs measure to another.

- Applied to two-dimensional spin glasses at zero temperature, the conjecture implies the conjecture that the spin glass system has a unique ground-state pair.

Partial uniqueness result

- Due to the disorder in the systems considered, it does not make sense to consider translation-invariant Gibbs measures. Instead, the following notion of translation-covariant Gibbs measures has been proposed.
- A measurable map ρ from the disorder variables η to the Gibbs measures of the η-disordered system is called a translation-covariant Gibbs measure if
$$\rho\big(\mathcal{T}_v(\eta)\big) = \mathcal{T}_v\big(\rho(\eta)\big)$$
for all vertices $v \in \mathbb{Z}^d$ (the translation $\mathcal{T}_v$ naturally extends to Gibbs measures).
- Compactness arguments (Aizenman-Wehr, Newman-Stein) show that translation-covariant Gibbs measures always exist for the disordered systems considered above (as barycenters of translation-covariant metastates).
- Theorem: For a disordered lattice system with compact state space (as discussed above) in dimension $d = 2$, at temperature $0 \leq T < \infty$ and disorder strength $\lambda > 0$, it holds that η-almost surely, for all vertices $v \in \mathbb{Z}^2$, the value of
$$\langle f\big(\mathcal{T}_v(\sigma)\big)\rangle_{\rho(\eta)}$$
is the same for all translation-covariant Gibbs measures ρ.
- Corollary: For the two-dimensional spin glass model at zero temperature, if there exists a translation-covariant extremal Gibbs measure then there is a unique translation-covariant Gibbs measure up to a global sign flip.

Proof sketch for compact state space

- Theorem recalled: For the above disordered systems with compact state space in two dimensions, at $0 \le T < \infty$ and $\lambda > 0$, there exists $C > 0$ so that for all $L \ge 2$,

$$\mathbb{E}\left(\sup_{\tau_1,\tau_2:\mathbb{Z}^2\to S}\left\|\frac{1}{L^2}\sum_{v\in\Lambda_L^2}\langle f\big(\mathcal{T}_v(\sigma)\big)\rangle_{\Lambda_L^2}^{\tau_1} - \langle f\big(\mathcal{T}_v(\sigma)\big)\rangle_{\Lambda_L^2}^{\tau_2}\right\|\right) \le \frac{C}{(\log\log L)^{\frac{1}{4}}}$$

- To simplify, assume $f(\sigma) = f(\sigma_{\mathbf{0}}) \in \mathbb{R}$ and fix $T > 0$. Write $Z_{T,\Lambda,\tau}^{\eta}$ for the partition function at temperature T, in a finite $\Lambda \subset \mathbb{Z}^2$ and with boundary conditions τ. Thus

$$Z_{T,\Lambda,\tau}^{\eta} := \int e^{-\frac{1}{T}H_\Lambda^\eta(\sigma)}\prod_{v\in\Lambda} d\kappa(\sigma_v)\prod_{v\in\Lambda^c}\delta_{\tau_v}(\sigma_v)$$

with $H_\Lambda^\eta(\sigma)$ the terms in the Hamiltonian $H^\eta(\sigma) = H(\sigma) - \lambda\sum_v \eta_v f\big(\mathcal{T}_v(\sigma)\big)$ depending on the spins in Λ. Let $F_\Lambda^\eta(\tau) := \frac{T}{|\Lambda|}\log Z_{T,\Lambda,\tau}^\eta$ be minus the free energy.

- Standard facts: 1) $F_\Lambda^\eta(\tau)$ is a convex function of η.
- 2) For each Λ: $\sup_{\tau_1,\tau_2}\left|F_\Lambda^\eta(\tau_1) - F_\Lambda^\eta(\tau_2)\right| \le \frac{C|\partial\Lambda|}{|\Lambda|}$.
- 3) Write $\eta = (\hat{\eta}_\Lambda, \eta_\Lambda^\perp)$ where $\hat{\eta}_\Lambda := \frac{1}{|\Lambda|}\sum_{v\in\Lambda}\eta_v$ and $\eta_{\Lambda,v}^\perp := \eta_v - \hat{\eta}_\Lambda$. Then $\frac{\partial}{\partial\hat{\eta}_\Lambda}F_\Lambda^{(\hat{\eta}_\Lambda,\eta_\Lambda^\perp)}(\tau) = \frac{\lambda}{|\Lambda|}\sum_v\langle f\big(\mathcal{T}_v(\sigma)\big)\rangle_\Lambda^\tau$, with the sum over terms involving spins in Λ

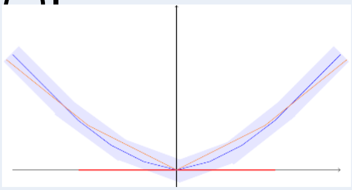
Proof sketch II

- Lemma: Let Λ satisfy $|\partial\Lambda| \le C\sqrt{|\Lambda|}$. Then for each $\delta > 0$,

$$\mathbb{P}\left(\sup_{\tau_1,\tau_2:\mathbb{Z}^d\to S}\left|\frac{\lambda}{|\Lambda|}\sum_v f\left(\mathcal{T}_v\left(\sigma^{\eta}_{\Lambda,\tau_1}\right)\right) - f\left(\mathcal{T}_v\left(\sigma^{\eta}_{\Lambda,\tau_2}\right)\right)\right| < 2\delta\right) \ge \exp\left(-\frac{C\lambda^2}{\delta^4}\right)$$

- Proof sketch: Claim: Let g: $\mathbb{R}\to\mathbb{R}$ be a convex 1-Lipschitz function. Set

$$N_r(g) := \{\, h: \mathbb{R}\to\mathbb{R} \text{ convex 1-Lipschitz } \mid \|h-g\|_\infty \le r \,\}.$$



Then for each $r, \delta > 0$.

$$\mathrm{Leb}(\{\, x\in\mathbb{R} \mid \exists h \in N_r(f), |h'(x) - g'(x)| \ge \delta \,\}) \le \frac{Cr}{\delta^2}$$

- Fix $\tau_0: \mathbb{Z}^2 \to S$ and let $g(x) := F_\Lambda^{(x,\eta_\Lambda^\perp)}(\tau_0)$. Then for all τ, $F_\Lambda^{(\cdot,\eta_\Lambda^\perp)}(\tau) \in N_{\frac{C|\partial\Lambda|}{|\Lambda|}}(g)$.

  Thus, the Claim implies that

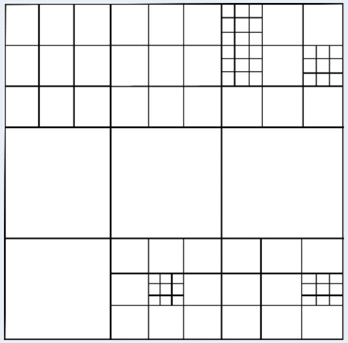
$$\mathrm{Leb}\left(\left\{x\in\mathbb{R}\;\middle|\; \exists\tau:\mathbb{Z}^2\to S,\ \left|\frac{\partial}{\partial\hat{\eta}_\Lambda} g_\Lambda^{(x,\eta_\Lambda^\perp)}(\tau) - \frac{\partial}{\partial\hat{\eta}_\Lambda} g_\Lambda^{(x,\eta_\Lambda^\perp)}(\tau_0)\right| \ge \delta\right\}\right) \le \frac{C\lambda|\partial\Lambda|}{|\Lambda|\delta^2} \le \frac{C\lambda}{\sqrt{|\Lambda|}\delta^2}$$

- Since $\hat{\eta}_\Lambda := \frac{1}{|\Lambda|}\sum_{v\in\Lambda}\eta_v$ is Gaussian with standard deviation $\frac{1}{\sqrt{|\Lambda|}}$ we conclude that

$$\mathbb{P}\left(\sup_{\tau:\mathbb{Z}^d\to S}\left|\frac{\lambda}{|\Lambda|}\sum_v f\left(\mathcal{T}_v\left(\sigma^{\eta}_{\Lambda,\tau}\right)\right) - f\left(\mathcal{T}_v\left(\sigma^{\eta}_{\Lambda,\tau_0}\right)\right)\right| < \delta\right) \ge \exp\left(-\frac{C\lambda^2}{\delta^4}\right)$$

which implies the lemma.

Proof sketch III



- Let $L \geq 2$. Call a set $\Lambda' \subset \Lambda_L$ ϵ-fluctuative if

$$\sup_{\tau_1,\tau_2:\mathbb{Z}^d\to S}\left|\frac{\lambda}{|\Lambda'|}\sum_v f\left(\mathcal{T}_v\left(\sigma^{\eta}_{\Lambda',\tau_1}\right)\right)-f\left(\mathcal{T}_v\left(\sigma^{\eta}_{\Lambda',\tau_2}\right)\right)\right|<\epsilon$$

- Perform a fractal percolation: Set $\delta := \frac{C\sqrt{\lambda}}{(\log\log L)^{\frac{1}{4}}}$ and $\mathrm{k} = C\lambda/\delta$.

  Partition Λ_L into k squares. Then partition each of these into k squares and so on until reaching squares of constant size. A square in this recursive partition is taken if it is 4δ-fluctuative and the squares containing it are not 4δ-fluctuative.

- Define $B := \{v \in \Lambda_L \mid v \text{ is not in a taken square}\}$. Then

$$\sup_{\tau_1,\tau_2:\mathbb{Z}^d\to S}\left|\frac{\lambda}{|\Lambda_L|}\sum_v f\left(\mathcal{T}_v\left(\sigma^{\eta}_{\Lambda_\mathrm{L},\tau_1}\right)\right)-f\left(\mathcal{T}_v\left(\sigma^{\eta}_{\Lambda_\mathrm{L},\tau_2}\right)\right)\right|\leq 4\delta+\frac{C|B|}{|\Lambda_L|}$$

- It remains to show that $\mathbb{P}(v \in B) \leq \delta$. Write $\Lambda_0(v) \supset \Lambda_1(v) \supset \Lambda_2(v) \supset \cdots$ for the partition squares containing v. Since $|\Lambda_{\ell+1}(v)| \leq c\delta|\Lambda_\ell(v)|/\lambda$, one concludes that

$$\{v \in B\} \subset \bigcap_\ell \{\Lambda_\ell(v) \setminus \Lambda_{\ell+1}(v) \text{ is not } 2\delta\text{−fluctuative}\}$$

- The events in the intersection are independent since the annuli are disjoint. The previous lemma bounds their probabilities, concluding the proof.

Non-compact case: Random-field random surfaces

- We now discuss the effect of disorder on systems with non-compact state space. Our focus is on random surface models.
- Let $(\eta_v)_{v\in\mathbb{Z}^d}$ be independent standard Gaussian random variables.
- A real-valued random-field random surface is the model on $\phi\colon \mathbb{Z}^d \to \mathbb{R}$ with Hamiltonian

$$H^\eta(\phi) = \sum_{u\sim v} V(\phi_u - \phi_v) - \lambda \sum_v \eta_v \phi_v$$

 where $\mathrm{V}\colon \mathbb{R} \to \mathbb{R}$ is a measurable even function termed the potential.
 The case $\mathrm{V}(x) = x^2$ is the real-valued random-field Gaussian free field.
- We also study the integer-valued random-field Gaussian free field which has the same Hamiltonian as above with $V(x) = x^2$ but restricts to $\phi\colon \mathbb{Z}^d \to \mathbb{Z}$.
- Our goal the localization/delocalization properties of these disordered surfaces.
- Without disorder: the gradient of these surfaces localizes in all dimensions $d \geq 1$.
 On Λ_L^d, real-valued surfaces delocalize with variance L when $d = 1$ and with variance $\log L$ when $d = 2$ while staying localized for $d \geq 3$.
 The integer-valued GFF behaves similarly except for a roughening transition when $d = 2$, from localized to logarithmic delocalization as the temperature increases.

Random-field random surfaces: results

- Theorem (Dario-Harel-P 2020+): Consider the real-valued random-field random surfaces above at all temperatures $0 \leq T < \infty$ and all disorder strengths $\lambda > 0$ on Λ_L^d with zero boundary conditions. Assume $0 < c_- \leq V'' \leq c_+ < \infty$. Then
  - Discrete Gradient: $\mathbb{E}\left(\left\langle \frac{1}{L^d} \sum_{\{\mathrm{u},\mathrm{v}\} \in E(\Lambda_L^d)} (\phi_\mathrm{u} - \phi_v)^2 \right\rangle\right) \approx \begin{cases} L & \mathrm{d} = 1 \\ \log L & d = 2 \\ 1 & d \geq 3 \end{cases}$
  - Height fluctuations: $\mathbb{E}(\langle \phi_\mathbf{0} \rangle^2) \approx \begin{cases} L^{4-d} & d = 1,2,3 \\ \log L & d = 4 \\ 1 & d \geq 5 \end{cases}$
- Theorem (Dario-Harel-P 2020+): The integer-valued random-field Gaussian free field, at all temperatures $0 \leq T < \infty$ and disorder strengths $\lambda > 0$, satisfies the gradient estimate above, and, when $d = 1,2$, satisfies

$$\mathbb{E}\left(\left\langle \frac{1}{L^d} \sum_{v \in \Lambda_L^d} \phi_v^2 \right\rangle\right) \approx L^{4-d}$$

  Additionally, this expectation is bounded in L in dimensions $d \geq 3$ at low temperatures and small disorder strength $\lambda > 0$.

Random-field random surfaces: previous results

- Bovier-Külske studied a random field Solid-On-Solid model in which the disorder enters differently from the way it is introduced here. They proved a certain form of delocalization in two dimensions (Bovier-Külske 1996) and localization in three and higher dimensions (Bovier-Külske 1994).

- Külske and Orlandi 2006 prove that for all deterministic fields η, a random surface with field η will delocalize with at least logarithmic variance in two dimensions, when the potential V satisfies $\sup V(x) < \infty$.

- Van Enter and Külske 2008 proved a form of delocalization for the gradients of the random-field random surface for a wide class of potentials in two dimensions. The result is non-quantitative.
  They further proved a lower bound on the rate of correlation decay for gradient Gibbs measures, when they exist, in three dimensions.

- Cotar and Külske proved the existence of translation-covariant gradient Gibbs measures for random-field random surfaces in dimensions $d \geq 3$ (Cotar and Külske 2012) and their uniqueness for each given expected tilt (Cotar and Külske 2015), for a large class of potentials.

- Later results: Dario 2023 (thermodynamic limit), Sakagawa 2023 (maximum).

Open questions

- For disordered systems with compact state space, improve the bounds on

$$\mathbb{E}\left(\sup_{\tau_1,\tau_2:\mathbb{Z}^d\to S}\left\|\frac{1}{L^d}\sum_{v\in\Lambda_L^d}\langle f(\mathcal{T}_v(\sigma))\rangle_{\Lambda_L^d}^{\tau_1}-\langle f(\mathcal{T}_v(\sigma))\rangle_{\Lambda_L^d}^{\tau_2}\right\|\right)$$

  If the sum is performed over a concentric box of half the size, does it decay exponentially fast with L in two dimensions at all T and $\lambda > 0$?

- Uniqueness conjecture: For two-dimensional disordered systems, for each $v \in \mathbb{Z}^2$, η-almost surely, the value of $\langle f(\mathcal{T}_v(\sigma))\rangle_\mu$ is the same for all Gibbs measures μ.

- Is there a Berezinskii-Kosterlitz-Thouless type transition as the disorder strength lowers (i.e., transition from exponential to power-law decay) for the random-field spin $O(n)$ models with $n = 2$ in dimensions $d = 3$ or $d = 4$? What about $n \geq 3$?

- What is the localization/delocalization behavior of the integer-valued random-field Gaussian free field in dimensions $d \geq 3$ at high disorder strength λ?
  Conjecture: Delocalization in dimension $d = 3$ and localization when $d \geq 5$.
  Thus we conjecture a roughening transition in the disorder strength for $d = 3$.